Isospectral Plane Domains

by
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Abstract

If two drums make the same sound, do they have the same shape? This question, in an idealized form, attracted the attention of mathematicians. It turned out that the answer to the question is no: there are some drums that sound the same but have different shapes. We call these special drum shapes isospectral non-congruent domains.

This thesis will exhibit such a pair of isospectral non-congruent plane domains, and prove that they are isospectral. It will also explain a more general technique for producing isospectral manifolds, and use it to construct isospectral surfaces.

Acknowledgments

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Introduction

In the 1960s, a certain question became well-known among mathematicians: If we know what sounds a drum makes when it is struck, can we determine the shape of the drumhead? In short, "Can one hear the shape of a drum?"

To convert this question into a math problem, we need to construct a mathematical model of a drum. Our drumhead will be a domain $D \in \mathbb{R}^2$. When the drum is struck, its skin will begin to oscillate up and down. We can represent the height of the drum at each point on its surface by a function $u : D \to \mathbb{R}$. This function will evolve in time according to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Since the drum skin is clamped to the rim at the edges of the drum, we also must have $u(x, y) = 0$ for (x, y) on the boundary of D.

As it vibrates, the drum will produce waves in the air that we hear as sounds. We will hear a pure tone if the drum oscillates at a single constant frequency everywhere – if $u(x, y, t)$ can be written as $u(x, y, t) = f(x, y)(\cos(\gamma t) + \sin(\gamma t))$ for some $\gamma \in \mathbb{R}$. This implies

$$-\gamma^2 u = c^2 \Delta u$$

$$\Delta u = -\left(\frac{\gamma}{c}\right)^2 u$$

Therefore, the pure tones that can be produced by a drum with a given shape $D \in \mathbb{R}^2$ are in one-to-one correspondence with the eigenvalues of Δ on D. The set of possible eigenvalues of Δ on D is referred to as the spectrum of D.

Some properties of the geometry of a plane domain D can be determined by its spectrum alone. For example, Weyl [19] derived the following asymptotic formula for the eigenvalues:

$$\lambda_m \sim \frac{4\pi m}{A},$$

where A is the area of the domain, and λ_m is the mth eigenvalue, where the eigenvalues are arranged in an increasing sequence. This formula implies that

$$A = \lim_{m \to \infty} \frac{4\pi m}{\lambda_m}$$

So the area of a domain is uniquely determined by its spectrum.

Given that the area can be deduced from the spectrum, as well as other geometric invariants, it's natural to wonder if the entire shape is so determined. This is the mathematical phrasing of "can one hear the shape of a drum?": can one determine a plane domain from its spectrum? Similar questions can be asked for general manifolds, and higher-dimensional analogues of Δ and the spectrum. Gel'fand conjectured that the spectrum of a Riemann surface determines the surface's metric up to isometry[3]. It turns out that both of these conjectures are false.

For plane domains, the problem was popularized among mathematicians by Mark Kac's 1966 article [12]. For general manifolds, it was proved that the spectrum does not determine the manifold by Milnor [14], who constructed a pair of 16-dimensional tori with the same spectrum that were not isometric. Several other examples of isospectral manifolds were found, but a breakthrough occurred when Sunada [18] discovered a general method for constructing such manifolds by considering the quotient of manifolds under finite group actions. This method was extended by Gordon, Webb, and Wolpert[9] to produce the first example of isospectral domains in the plane. Their proof was simplified by Buser, Conway, Doyle and Semmler [5], who constructed some particularly simple examples and gave an elementary proof of isospectrality based on "transplantation".

This thesis will present the proof of the existence of isospectral non-congruent planar domains in a comprehensible fashion. Section 1 explain some basic properties of the Laplacian operator. Section 2 will present Conway et al's "transplantation" proof. Section 3 will explain Sunada's method and exhibit some examples of isospectral surfaces created using this method. Finally, section 4 will show how Sunada's method can ultimately be used to derive the elementary proof by transplantation.

1 The Laplacian and its Properties

In this section, we will define the Laplacian and its spectrum on a plane domain. We will define what it means for two planar domains to be isospectral, and prove some properties of the Laplacian and its eigenvalues.

Definition 1. *The Laplacian, or Laplace operator, is an 2nd-order differ-*

ential operator Δ on twice-differentiable functions on $\mathbb{R}^n$ such that

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

The Laplacian can be extended to act on Riemannian manifolds using the equivalent definition $\Delta f = \nabla \cdot \nabla f$.

Definition 2. *Given some bounded domain $D \subset \mathbb{R}^n$, a nonzero function f is an eigenfunction of the Laplacian on the domain if it satisfies $\Delta f = -\lambda f$ for all $x \in D$. The value λ is referred to as an eigenvalue of the Laplacian on the domain.*

If the manifold has boundary then eigenfunctions are required to satisfy certain boundary conditions. For domains $D \in \mathbb{R}^n$ the typical boundary conditions are:

1. Dirichlet. $f(x) = 0$ for all $x \in \partial D$, where ∂D is the boundary of D.
2. Neumann. $\nabla f(x) \cdot n = 0$ for all $x \in \partial D$, where n is the normal to the boundary.

In this report we will focus on the Dirichlet boundary conditions.

The linear combination of eigenfunctions with a given eigenvalue will be another eigenfunction with the same eigenvalue; they form a vector space. We say that an eigenvalue λ has geometric multiplicity m if the corresponding vector space of eigenfunctions has dimension m.

It can be shown [8] for domains with piecewise smooth boundary that the multiplicities of the eigenvalues are finite, that the set of eigenvalues is infinite, has no limit points, and consists of positive real numbers. The eigenfunctions also form an orthonormal basis for $L^2(D)$. We will present a proof sketch of this later. Thus, we can arrange the eigenvalues in an infinite sequence

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$$

in which we repeat each value according to its multiplicity (λ_1 always has multiplicity 1). We refer to this sequence as the **spectrum** of D. We say that two domains are isospectral if they have the same spectrum. The problem we are considering is how to produce domains that are isospectral but not isometric.

In the remainder of this section, we'll first present some examples of domains and their spectra, and then prove some of the properties of the eigenvalues we claimed above. We'll also present a method for extending eigenfunctions to domains beyond their original domain of decomposition: the Reflection Principle. This principle will be needed for the proof of isospectrality by transplantation.

1.1 Eigenvalues of the Line

As a simple example of a domain for which we can determine the spectrum, consider the 1 dimensional line segment $[0,1] \subset \mathbb{R}$. Here, the problem of finding the Dirichlet eigenvalues is equivalent to solving the differential equation

$$\frac{d^2 f}{dx^2} = -\lambda f$$

subject to $f(0) = 0$ and $f(1) = 0$.

The ordinary differential equation $\frac{d^2 f}{dx^2} = -\lambda f$ has solutions:

- $f(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ for $\lambda < 0$. Given $f(0) = 0$, we have $c_1 = -c_2$, but this implies that $f(b) = c_1 e^{\sqrt{-\lambda}} - c_1 e^{-\sqrt{-\lambda}}$ will not be zero, as $e^{\sqrt{-\lambda}} \neq e^{-\sqrt{-\lambda}}$. Hence there are no Dirichlet eigenvalues for $\lambda < 0$.

- $f(x) = ax + d$ for $\lambda = 0$. We have $f(0) = d = 0$, so $f(1) = a = 0$, which implies $a = 0$. Therefore there are no non-trivial eigenfunctions with eigenvalue 0.

- $f(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ for $\lambda > 0$. In this case, we have $f(0) = c_1 = 0$, so $f(1) = c_2 \sin(\sqrt{\lambda}) = 0$. Assuming the solution is not trivial, this implies $\sin(\sqrt{\lambda}) = 0$, so $\sqrt{\lambda} = \pi k$ for some $k \in \mathbb{Z}$. Thus $\lambda = (\pi k)^2$ for some $k \in \mathbb{Z}$.

So then, for an interval of length 1, the spectrum of the Laplacian is given by

$$\{\lambda_i = \{(\pi i)^2 | i = 1, 2, ...\}$$

1.2 Eigenvalues of the Disk

A slightly more challenging example is given by the planar disk of radius L,

$$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq L^2\}$$

This exposition is based on [11]

We can simplify the problem by switching to polar coordinates. Our region is now $\{(r, \theta) | r \in [0, L], \theta \in [0, 2\pi)\}$. In polar coordinates the Laplacian takes the form

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

So eigenfunctions will satisfy

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = -\lambda f$$

with $f(L, \theta) = 0$.

If we separate variables and write $f(r, \theta) = R(r)\Theta(\theta)$, the above equation can be written

$$(R'' + \frac{1}{r}R' + \lambda R)\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\frac{r^2 R'' + rR' + \lambda r^2 R}{R} + \frac{\Theta''}{\Theta} = 0$$

$$\frac{r^2 R'' + rR' + \lambda r^2 R}{R} = -\frac{\Theta''}{\Theta}$$

Since the left-side of this equation is dependent only on R and the right side only dependent on Θ, we have $\frac{\Theta''}{\Theta} = -c$. Now, Θ is 2π-periodic, so it has the form $A\cos(m\theta) + B\sin(n\theta)$. $\frac{\Theta''}{\Theta} = -c$ everywhere, so $\frac{\Theta''(0)}{\Theta(0)} = -m^2$ implies that $c = -m^2$. So our equation for R becomes

$$r^2 R'' + rR' + (\lambda r^2 - m^2)R = 0$$

Substitute $q = \sqrt{\lambda}r$ and this becomes

$$q^2 R'' + qR' + (q^2 - m^2)R = 0$$

Solutions to this equation are called Bessel functions, $J_m(x)$. We have $R(r) = J_m(\sqrt{\lambda}r)$. Using our boundary conditions, we have $R(L) = 0$. So $J_m(\sqrt{\lambda}L) = 0$. The functions $J_m(x)$ have a discrete collection of zeros; let z_{mn} denote the nth zero of J_m, and let N_m be the number of zeros of J_m. Then the allowable values of lambda are

$$\{\lambda_{mn} = \left(\frac{z_{mn}}{L}\right)^2 | m \in \mathbb{N}, 1 \leq n \leq N_m\}$$

1.3 Properties of Eigenvalues

We claimed above several properties of the eigenvalues of the Laplacian. Here, we will sketch a proof of these properties.

First, positivity. This proof of positivity is found in [13]. When we say the eigenvalues are positive, we mean that for all non-trivial solutions of $\lambda f = -\Delta f$ we have $\lambda > 0$.

Consider the integral

$$\lambda \int_D f^2 d\vec{x} = -\int_D (\Delta f) f d\vec{x}$$

$$= -\left(\int_{\partial D} (\nabla f \cdot \vec{n}) f - \int_D |\nabla f|^2 \right)$$

$$= \int_D |\nabla f|^2 d\vec{x}$$

where we write $dx_1 dx_2 ... dx_2 = d\vec{x}$, and $\vec{n}$ is the normal vector to the boundary of D.

So in total we have

$$\lambda \int_D f^2 d\vec{x} = \int_D |\nabla f|^2 d\vec{x}.$$

Observe that $\int_D |\nabla f|^2 d\vec{x}$ cannot be zero because this would mean $\frac{\partial f}{\partial x_i}$ is zero everywhere for all i. Since f is zero on the boundary, this would mean f is identically zero. However, by definition $f(x) = 0$ is not an eigenfunction. Since f^2 and $|\nabla f|^2$ are positive, this implies that λ is positive.

We also wish to prove that each eigenspace is finite dimensional, that there are infinitely many eigenvalues, and that the set of eigenvalues has no limit points. The proof is rather technical, but we can give an outline.

We can extend the Laplacian to the space of L^2-integrable functions on our domain, $L^2(D)$. This space can be endowed with an inner product, $\langle f, g \rangle = \int_D fg$. This makes it a Hilbert Space, which is an inner product space that is complete in the induced metric [17].

There is a useful theorem for understanding the eigenvalues of operators acting on some Hilbert space H, called the spectral theorem for compact self-adjoint operators. This theorem can be applied if the operator T is:

- compact, which means that the closure $\overline{T(B)}$ of the image of a bounded subset $B \in H$ is compact.

- self-adjoint, which means that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$.

given these conditions, we can state the spectral theorem. [15]

Theorem 1. *Suppose T is a compact, self-adjoint operator on H. Then there exists a system of orthonormal eigenvectors $\psi_1, \psi_2, \psi_3...$ with corresponding real eigenvalues $\lambda_1, \lambda_2, \lambda_3, ...$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq ...$, and*

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, \psi_k\rangle \psi_k$$

If H is separable(has a countable orthonormal basis), the eigenvectors ψ_i form such an orthonormal basis for H. If the set (λ_n) is infinite, then (λ_n) approaches zero as n approaches infinity.

Unfortunately, the Laplace operator is not bounded, so we cannot apply the spectral theorem directly. To apply the spectral theorem, we consider another operator Δ^{-1} on the space of L^2 functions, the inverse Laplacian, such that

$$\Delta f = g \Leftrightarrow f = \Delta^{-1} g$$

It can be shown [8] that this operator Δ^{-1} exists and is compact, self-adjoint, and positive. Now consider the eigenvalues γ of Δ^{-1}:

$$\Delta^{-1} g = -\gamma g$$

$$g = -\gamma \Delta g$$

$$\Delta g = -\frac{1}{\gamma} g$$

So we see that the eigenvalues of Δ are inverse to those of Δ^{-1}. Since we proved above that the eigenvalues of Δ are positive, the eigenvalues of Δ^{-1} must be positive as well. By the spectral theorem, the eigenvalues γ_n of Δ^{-1} can be arranged in a sequence

$$\gamma_1 > \gamma_2 \geq \gamma_3 \geq ...$$

which approaches 0 as n approaches ∞. Therefore, the corresponding eigenvalues of the Laplace operator, $\lambda_n = \frac{1}{\gamma_n}$, can be arranged in a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$$

which approaches infinity as n approaches infinity. This was our claim.

1.4 The Reflection Principle

The reflection principle allows us to extend eigenfunctions of the Laplacian defined on a subset of $\mathbb{R}^n$ to a wider subset of $\mathbb{R}^n$. In particular, if we have some domain D which is symmetric about a hyperplane P, we can consider the subdomains D_+ and D_- lying on either side of P. Then an eigenfunction of the Laplacian defined on D_+ can be extended to one defined on D [7]. This property will be crucial for the proof of isospectrality by the transplantation method.

The reflection principle for eigenfunctions can easily be proved using the reflection principle for harmonic functions, which are functions satisfying $\Delta g = 0$ everywhere. This reflection principle can in turn by proved using complex-analytic methods[16], but we will present it without proof:

Theorem 2. ***Reflection Principle for Harmonic Functions*** *Suppose D is a bounded domain in $\mathbb{R}^n$ which is symmetric about a $(n-1)$-plane P. Denote by D_+ and D_- the subdomains lying on either side of P. Denote by x^* the reflection of a point x in the plane P. If we have a harmonic function f_+ defined on D_+ such that $f_+(p) = 0$ for $p \in P$, then the function $f(x)$ defined by*

$$f(x) = \begin{cases} f_+(x) & x \in D_+ \\ -f_+(x^*) & x \in D_- \end{cases}$$

is a harmonic function on D.

Using this theorem, we can easily prove the corresponding reflection principle for eigenfunctions of the Laplacian:

Theorem 3. ***Reflection Principle for Eigenfunctions*** *Suppose D is a bounded domain in $\mathbb{R}^n$ which is symmetric about a $(n-1)$-plane P. Denote by D_+ and D_- the subdomains lying on either side of P. Denote by x^* the reflection of a point x in the plane P. If we have a eigenfunction of the Laplacian f_+ with eigenvalue λ defined on D_+ such that $f_+(p) = 0$ for $p \in P$, then the function $f(x)$ defined by*

$$f(x) = \begin{cases} f_+(x) & x \in D_+ \\ -f_+(x^*) & x \in D_- \end{cases}$$

is an eigenfunction of the Laplacian defined on D.

Proof. Since the Laplacian is invariant under rotations and reflections, without loss of generality we can let the hyperplane be $x_1 = 0$, and let D_+ lie in $x_1 > 0$. Define a function g_+ in $D_+ \times \mathbb{R}$ by $g_+(x, a) = f_+(x)c(a)$, where

$$c(a) = \begin{cases} \exp(a\sqrt{-\lambda}) & \lambda \leq 0 \\ \cos(a\sqrt{\lambda}) & \lambda \geq 0 \end{cases}$$

Since f is an eigenfunction of Δ on D_+, we have

$$\Delta g_+ = \Delta f_+ c(a) + f_+ \frac{\partial^2(c)}{\partial a^2} = -\lambda f_+ c + \lambda f_+ c = 0$$

Therefore, g_+ satisfies the Laplace equation $\Delta g = 0$ on $D_+ \times \mathbb{R}$. Now, note that $D_+ \times \mathbb{R}$ is a sub-domain of $\mathbb{R}^{n+1}$. The set of points such that $x_1 = 0$ is an n-plane in $\mathbb{R}^{n+1}$, and D_+, D_- are symmetric about this plane. By the reflection principle for harmonic functions we can extend g_+ to a function g defined on $D \times \mathbb{R}$, namely

$$g(x,a) = \begin{cases} g_+(x,a) & x \in D_+ \times \mathbb{R} \\ -g_+(x^*,a) & x \in D_- \times \mathbb{R} \\ 0 & x \in P \times \mathbb{R} \end{cases}$$

Evidently, we have $g(x,a) = f(x)c(a)$. Also,

$$\Delta g = \Delta f c(a) + f\frac{\partial^2(c)}{\partial a^2} = 0$$

At $a = 0$, this simplifies to

$$\Delta f + \lambda f = 0$$

which implies that f is an eigenfunction of the Laplacian with eigenvalue λ on D.

$\square$

2 The Method of Transplantation

In [5], a simple method is given for demonstrating that certain domains are isospectral. Here, we will illustrate this method, using an example from the cited paper.

Consider two domains obtained by pasting together 7 congruent triangles along sides of equal length, depicted in figure 1. The triangle is required to have all angles less than $\frac{\pi}{2}$, so that no copies overlap in the final domain. If the triangle is scalene(all sides having different length), then the produced domains will be non-congruent. To see this, note that in each domain, the central triangle is distinguished by having four triangles around each vertex. This proves that any mapping between the two domains must send one central triangle to the other. But if they are scalene, there is only one such mapping(translation), and the pattern of identifications of the sides shows that this will not be a congruence between the two domains.

So, we have two non-congruent domains. We wish to prove that they are isospectral. To do this, we will present a map Φ from the set of functions on

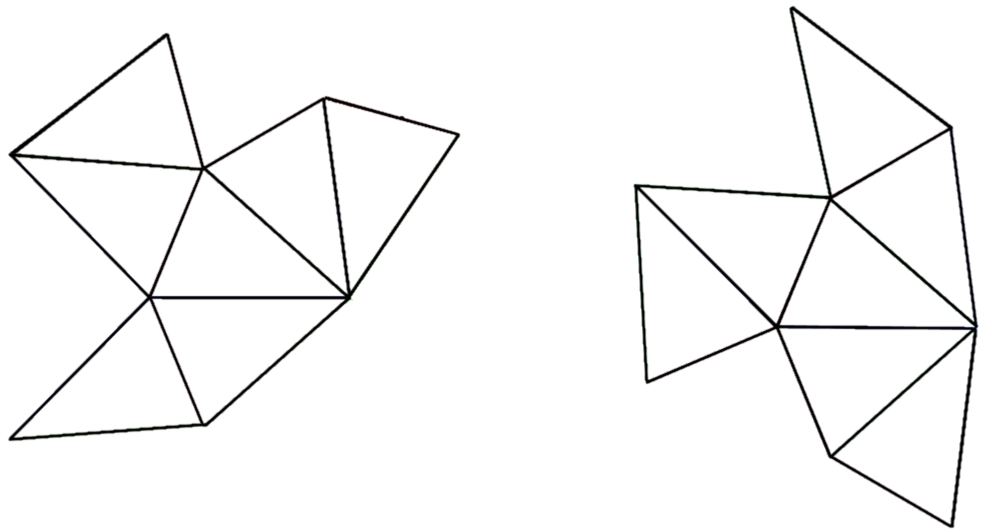


Figure 1: Our two isospectral domains

A to the set of functions on B, such that an eigenfunction with eigenvalue λ on A is sent to another such function on B. We will also exhibit a map from functions on B to functions on A with the same property. The existence of these maps show that A and B have the same spectrum.

To explain the mapping, we will number the triangular subdomains of A by A_0, A_1,...,A_6, and number the triangular subdomains of B by B_0, B_1, , ..., B_6. Assume we have an eigenfunction f defined on A with eigenvalue λ. Denote the restriction of f to the subdomain A_i by f_i.

Note that for each pair of triangular domains A_i,B_j, there is a unique isometry τ_{ij} sending A_i to B_j. We can use this isometry to map the function f_i to a domain B_k in a unique fashion. In particular we can define a function on B_k by $f_i \circ \tau_{ik}^{-1}$.

Using these isometries, we build an eigenfunction g on B using the scheme illustrated in figure 2. In the central triangle B_0, we define the restriction $g_0 = f_1 \circ \tau_{10}^{-1} + f_2 \circ \tau_{20}^{-1} + f_4 \circ \tau_{40}^{-1}$, and similarly for the other triangles. Since each τ_{ij}^{-1} is an isometry, it is clear that the resulting function will locally be an eigenfunction of Δ with eigenvalue λ. We just have to check that it remains smooth across the boundaries of triangular subdomains, so we can take second derivatives, and goes to zero at the boundary

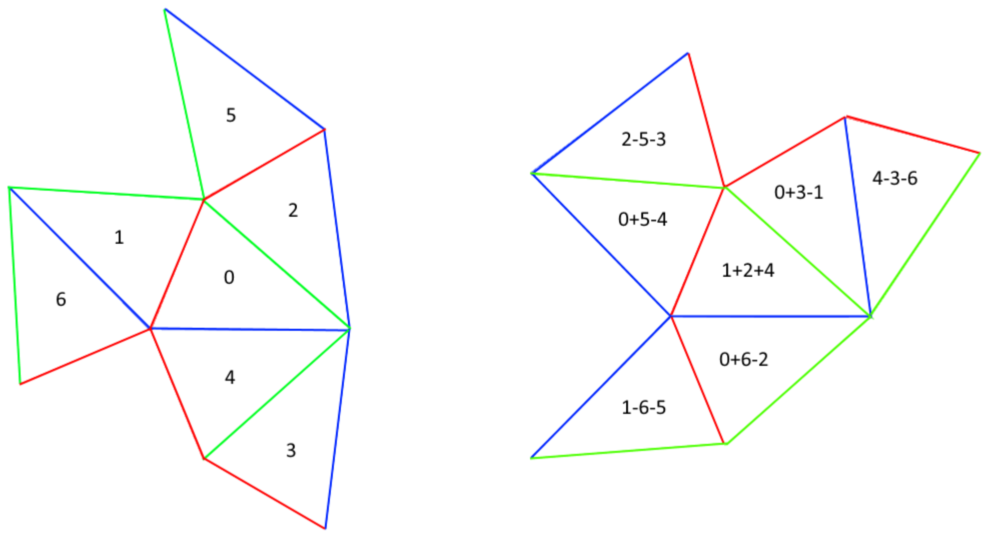


Figure 2: The mapping Φ from functions on A to functions on B

of B.

To see why the functions will be continuous along the internal boundaries, we consider the blue edge boundary of the central triangle of B. On one side of the boundary, we have assigned (leaving out τ_{ij} for clarity) $f_1+f_2+f_4$, on the other, $f_0+f_6-f_2$. Examining these triangles in A, we see that f_1 continues to f_6, f_2 continues to $-f_2$ by the reflection principle, and f_4 continues to f_0. This is shown in figure 3.

There is also continuity across the red edge of the triangle just below the center. In figure 4 we see that on one side we have $f_0+f_6-f_2$, on the other $f_1-f_6-f_5$. Once again, examining A, we see that, across the red edge, f_0 continues to f_1, f_6 to $-f_6$ (reflection principle), and $-f_2$ to $-f_5$. There is continuity along this edge as well.

Now let us consider external boundary edges. Look at the highlighted green edge shown in figure 5 on the right. Along this edge, the function $g_i = f_6 - f_2 + f_0$. But since f_6 goes to zero along the green edge, and $f_0 = f_2$ here, g_i will be zero along the boundary, as required. Similarly, along the blue edge of the bottommost triangle, we have $f_1 - f_5 - f_6$. Along this edge, f_1 continues to f_6, so f_1 and$-f_6$ cancel. f_5 goes to zero along the blue edge. So g_i will be zero along this boundary. Considering the green

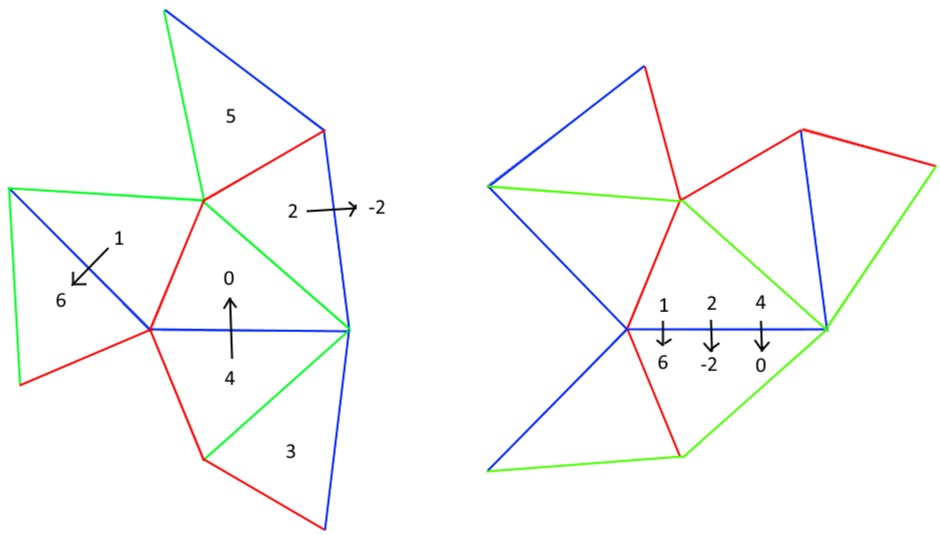


Figure 3: Crossing a blue edge

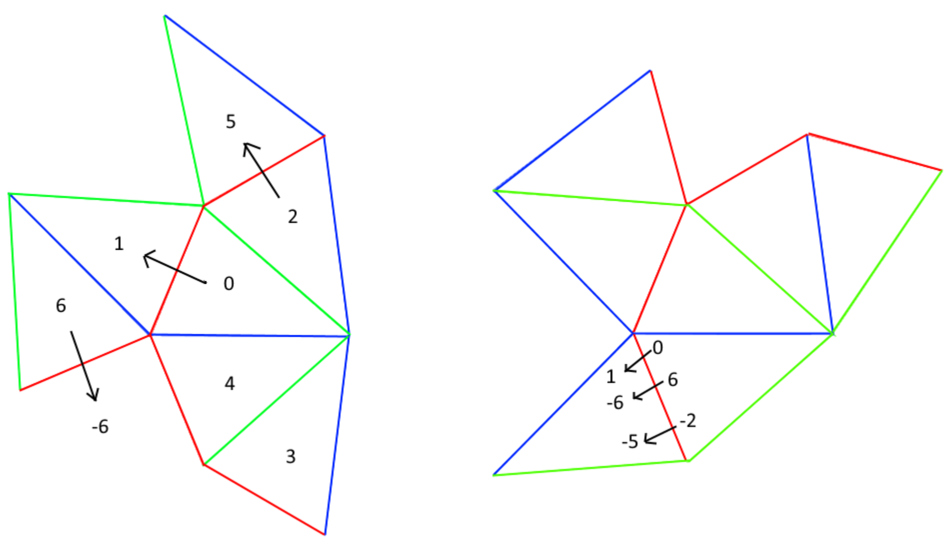


Figure 4: Crossing a red edge

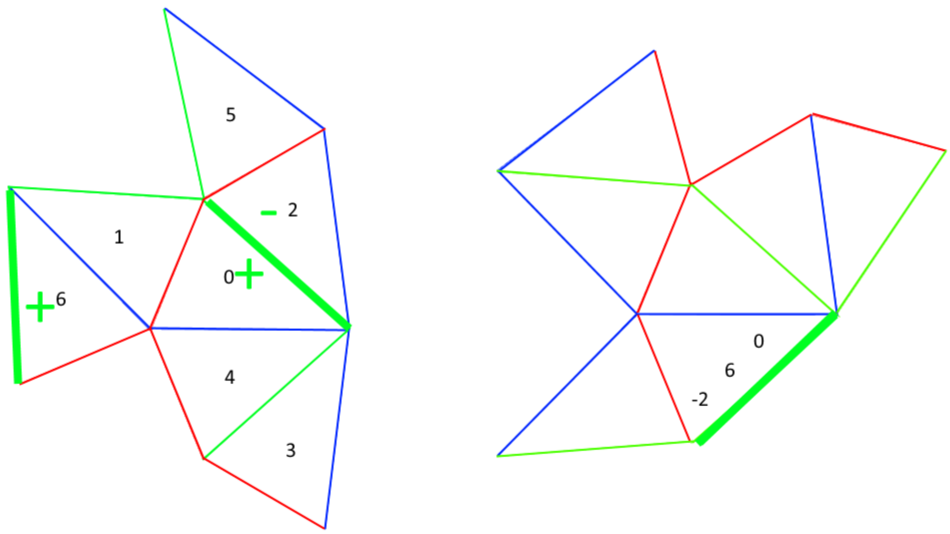


Figure 5: Boundary conditions

edge of the bottom-most triangle, we see that f_1, f_5 and f_6 all go to zero along the green edge.

Thus we have verified our transplanted function satisfies internal and external boundary conditions on the bottom "leg" of our domain. We can verify all other boundaries by using the permutation

$$1 \to 2 \to 4 \to 1$$

$$3 \to 6 \to 5 \to 3$$

which, it can easily be seen, cyclically permutes the "legs" of both domain. So our proof of satisfying boundary conditions for the bottom "leg" works for all others, just with the labels of functions cyclically permuted.

So, we see that this is a well-defined map between the eigenspace of A with eigenvalue λ and the eigenspace of B with eigenvalue λ. To show that it is an injection, we must prove that it is non-singular, i.e. that no non-zero functions are mapped to zero. This can easily be seen: if we have a non-zero function defined on domain A, there must be at least one point in A at which this function is not zero. Consider the 7 points in A and the 7 points in B which are the images of this point under the unique congruences between the triangular subdomains. Φ induces a linear map between the values of

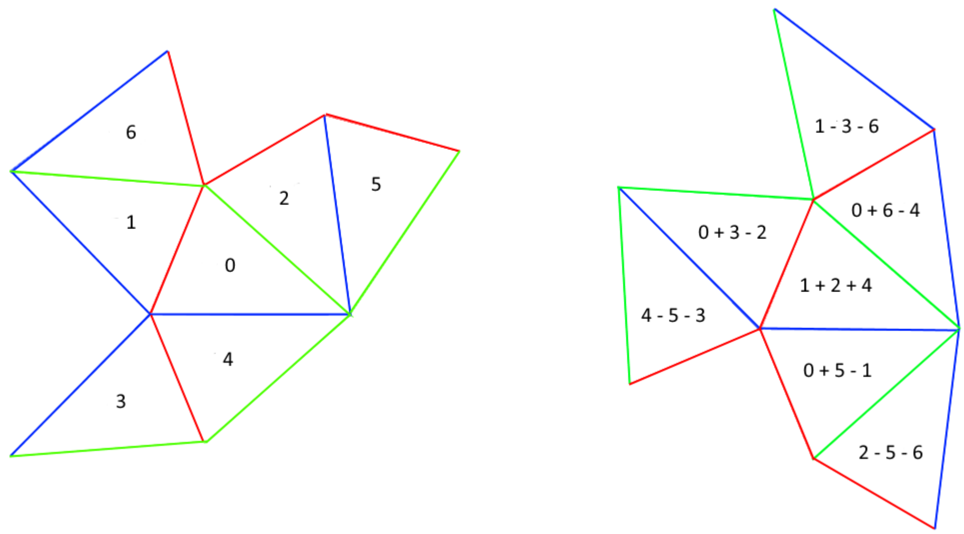


Figure 6: Mapping from functions on B to functions on A

the function at these 7 points in A and the values at these 7 points in B. This linear map can be represented by the matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

It can easily be checked that this matrix is non-singular. So the transplanted function on B will be non-zero at least one point, proving that Φ is non-singular.

To complete the proof that A and B are isospectral, we must exhibit an injective mapping Ψ from eigenspaces of B to eigenspaces of A. The pattern for such a mapping is shown in figure 6. The proof that it is injective is exactly analogous to the proof that Φ was injective.

The reader may at this point be satisfied that the two domains are isospectral, but be curious as to how anyone would think of the above mappings. There is a simple way. Having labelled our domains in A 0,1,...6, begin by placing "1" (for instance) in the central triangle of B. From there, we see that 1 continues to -1 across a green edge, 0 across a red edge, and 6 across a blue edge. So we place those numbers in their corresponding triangles in B. Then, examining the triangle below the centre in B, we have 0. 0 continues across a green edge to 2, so for this green edge boundary in B to be zero, we must have -2 in this domain. We proceed in this way, adding transplanted functions across boundaries, until all functions "close up" and our mapping is complete.

This doesn't explain *why* the functions should "close up", or why these particular domains work. To understand that, we must first understand an older method for proving isospectrality, Sunada's Method.

3 Sunada's Method

The above method of proving isospectrality was not arrived at out of nowhere. It was based on an older method for proving isospectrality of manifolds. In this section, we will first prove Sunada's theorem, which asserts the isospectrality of certain quotients of manifolds by the action of subgroups. This theorem uses the notion of a covering; for an explanation of this concept see [10]. Next, we will present some examples of groups satisfying the conditions of the theorem. Lastly, we will use Sunada's theorem to prove the isospectrality of certain surfaces.

3.1 Sunada's Theorem

Consider a finite group G and two subgroups H_1 and H_2. The triple (G, H_1, H_2) is said to satisfy the Sunada condition if for each conjugacy class G_i of G, $|G_i \cap H_1| = |G_i \cap H_2|$.

Theorem 4. *Let $\pi : M \to M_0$ be a normal finite Riemannian covering with deck group G. Let M_1 and M_2 be the coverings corresponding to subgroups H_1 and H_2. If (G, H_1, H_2) satisfy the Sunada condition, then M_1 and M_2 are isospectral.*

Proof. (after [1]):

Consider the spectrum of M_1. First, note that every eigenfunction of M_1 can be lifted to one of M. So the spectrum of M_1 is a subset of that of M. To determine the spectrum of M_1, it suffices to determine the multiplicity of each eigenvalue of M in M_1.

Each function on M that is invariant under the action of H_1 can be pushed forward to one on M_1. In fact, functions on M_1 are in one-to-one correspondence with those on M that are invariant under H_1.

We can project any function on M to one that is invariant under H_1 with the map P:

$$Pf = \frac{1}{|H_1|} \sum_{h \in H_1} T_h(f)$$

where T_h is the push-forward of f under the action of h on M. That is $T_h(f)$ is the composition $f \circ h$, where we interpret h as a isometry of M.

Since P is a projection, the dimension of each of the eigenspaces of the Laplacian will be equal to the trace of P. This is

$$Tr(P) = \frac{1}{|H_1|} \sum_{h \in H_1} Tr(T_h)$$

The trace is preserved by conjugation, so

$$Tr(P) = \frac{1}{|H_1|} \sum_{G_i} |G_i \cap H_1| Tr(T_{g_i}),$$

where the G_i's range over the conjugacy classes of their members, and each g_i is a member of G_i. This shows that the multiplicity of each eigenvalue depends only on the numbers $|G_i \cap H_j|$, demonstrating Sunada's Theorem.

$\square$

3.2 Example of a Sunada Triple

A useful example of a Sunada triple is found in $SL(n,q)$, the group of n by n matrices with entries in $\mathbb{F}_q$ with determinant 1. The following example was given in [2].

The two subgroups H_1, H_2 are given by matrices of the form

$$H_1 = \begin{bmatrix} * & * & * & & * \\ 0 & * & * & & * \\ 0 & * & * & & * \\ & & & & \\ 0 & * & & & * \end{bmatrix} \quad H_2 = \begin{bmatrix} * & 0 & 0 & & 0 \\ * & * & * & & * \\ * & * & * & & * \\ & & & & \\ * & * & & & * \end{bmatrix}$$

which are the elements stabilizing the line $(t, 0, ..., 0)$ and the $(n-1)$-plane $(0, x_2, ..., x_n)$ respectively. H_1 and H_2 are easily seen to be subgroups.

We will prove that the subgroups H_1 and H_2 are not conjugate, but do satisfy the Sunada condition.

Claim 1. *H_1 and H_2 are not conjugate.*

Proof. Assume H_1 and H_2 are conjugate, that is, there is some $g \in SL(n,q)$ such that $H_2 = gH_1g^{-1}$. Denote by e_1 the vector $(10...0)$. Consider the vector ge_1. Since e_1 is an eigenvector of each element of H_1, for each element $h_2 \in H_2$ we have

$$h_2ge_1 = gh_1g^{-1}ge_1 = gh_1e_1 = ge_1$$

which implies that ge_1 is an eigenvector of each element of H_2 with eigenvalue 1. We will prove that this is impossible, by exhibiting for each non-zero vector $v \in \mathbb{F}_q^n$ an element of H_2 which does not leave the vector invariant.

Consider our non-zero vector $v = (v_1 v_2 ... v_n)$. At least one element v_k, with index i must be non-zero. Choose some arbitrary other index l. Then the matrix Q defined by

$$Q_{kl} = 1$$

$$Q_{ij} = \delta_{ij}$$

for $i, j \neq k, l$ does not leave v invariant. Hence there cannot be any vectors which are fixed by all members of H_2, contradicting our conclusion above.

□

Claim 2. *H_1 and H_2 satisfy the Sunada condition.*

To prove this, we will study the intersection of an arbitrary conjugacy class with H_1. But first, another lemma:

Lemma 1. *Let ω be an element of $\mathbb{F}_q^n$. Let $V \subset \mathbb{F}_q^n$ be a subspace of dimension d. The number of transformations in $SL(n,q)$ mapping ω to an element of V is*

$$(q^d - 1)(q^n - q)(q^n - q^2)...(q^n - q^{n-1})$$

Proof. Choose a basis of $\mathbb{F}_q^n$ with ω as its first element. An element of $SL(n,q)$ is determined by the images of each element of this basis. There are $q^d - 1$ choices of non-zero vectors in $\mathbb{F}_q^n$ to send ω to. From there, each basis vector can be sent to any vector not in the subspace spanned by previous images. The number of such vectors will be $q^n - q^k$, where q^k is the number of basis vectors already assigned images. This in total gives us $(q^d - 1)\prod_k = 1^{n-1} q^n - q^k$ possible transformations, as desired.

□

Note in particular that this is only dependent on the dimension d of the subspace.

Lemma 2. *Given an element $g \in SL(n,q)$, the number of conjugates of g that are in H_1 depends only on i) the number of eigenspaces of g and their dimensionality, and ii) the size of the centralizer of g.*

Proof. Let $W = \{h_1 \in H_1 | h_1 = fhf^{-1}$ for some $f \in SL(n,q)\}$ A conjugate fgf^{-1} of g will intersect H_1 if and only if e_1 is an eigenvector of this conjugate – which implies that $f^{-1}e_1$ is an eigenvector of g. Equivalently, we can say that f^{-1} maps e_1 to one of the eigenspaces of g. Assume g has k eigenspaces $V_1, ..., V_k$ with eigenvalues $\lambda_1, ..., \lambda_k$. Denote by Ω_i the set of elements f of $SL(n,q)$ such that $f^{-1}e_1 \in v_i$ for some i, and denote by W_i the set of conjugates of g that have e_1 as an eigenvector with eigenvalue λ_i. Note $W = \cup W_i$. Note also that by the above lemma $|\Omega_i|$ is only dependent on the dimension of V_i Each conjugate of g can be written fgf^{-1} for some element f of a unique Ω_i. (Unique because each eigenspace V_i has associated with it an eigenvalue λ_i, and so $fgf^{-1}e_1 = \lambda_i e_1$.) Two elements q, j of Ω_i produce the same element of H_1 when conjugating g if $qgq^{-1} = jgj^{-1}$. Equivalently, $j^{-1}qgq^{-1}j = g$ – or, $j^{-1}q$ commutes with g. Consider the centralizer, $C(g)$ of g, the subgroup of elements of $SL(n,q)$ that commute with g. This subgroup has a well-defined action on Ω_i defined by

$$C(g) \times \Omega_i \to \Omega_i$$

$$h \times f \mapsto fh^{-1}$$

To see that this is well-defined, recall that Ω_i was defined as the set of elements such f that $f^{-1}e_1 \in V_k$ for some k Now, $(fh^{-1})^{-1} = hf^{-1}$

$$hf^{-1}e_1 = hV_k$$

Since h commutes with g, it must leave the eigenspaces of g fixed by the simultaneous diagonalization theorem. Therefore, $fh^{-1} \in \Omega_i$. The action is also free because it is a group action. It is clear that two elements of Ω_i will map to the same element of H_1 if and only if they are in the same orbit under the action of $C(g)$. So the map from the quotient

$$\Omega_i/C(g) \to V_i$$

is injective. Therefore,

$$|W| = \sum \frac{|\Omega_i|}{|C(g)|}$$

which is only dependent on the number/dimensionality of eigenspaces of g and the size of its centralizer.

□

We have established that the number of conjugates of g that are in H_1 depends only on the number of eigenvectors of g. Now, consider the automorphism A of $SL(n, q)$ defined by

$$Ag = \left(g^T\right)^{-1}$$

This automorphism sends H_1 to H_2. So the number of times that the conjugates of g intersect H_2 is equal to the number of times that the conjugates of $(g^T)^{-1}$ intersect H_1. But since taking the inverse and transpose leave the number of eigenvectors invariant, along with the number of elements in the centralizer, this is in turn equal to the number of times that g intersects H_1. So H_1 and H_2 satisfy the Sunada condition.

3.3 Constructing Surfaces via Relative Cayley Graphs

So we've obtained some subgroups satisfying Sunada's condition. To actually construct isospectral manifolds, we need to find a manifold on which G acts. There is a generic method, first used by Buser in [4], to construct a surface on which G acts. It involves the Cayley graphs of the groups.

First, some definitions. Given a group G, and a set of generators $\{g_i\}$ of the group, we can construct a (left) Cayley graph of the group. The graph's vertices correspond to elements of the group G, and edges correspond to the action of the generators of the group on group elements. That is, for each generator g_i, there is a labelled, directed arrow from an element x to y if $g_i x = y$. It is clear that each vertex will have one ingoing and one outgoing arrow for each generator, and that the graph will be connected.

As an example, consider the dihedral group of order 6, defined by:

$$\langle a, b | a^3, b^2, \rangle.$$

The Cayley graph of this group with respect to the generators a,b is shown in figure 7.

Another notion we will find useful is that of a relative Cayley graph. This can be constructed from a group G, a generating set for that group, and some subgroup H of G. In a relative Cayley graph, vertices correspond to cosets gH, and edges correspond to the natural action of the generators on cosets. For each generator g_i, there is a labelled, directed arrow from an element xH to yH if $g_i xH = yH$. For example, again consider D_6, with subgroup $H = \{1, b\}$, and generators a,b. This relative Cayley graph is shown in figure 8.

There is a simple way of constructing a surface from a Cayley graph(or relative Cayley graph). It requires a "building block", which is a surface with one marked boundary component for each generator and generator inverse g_i. Our surface consists of one copy of the building block for each vertex of the Cayley graph, with two building blocks glued together along their marked boundary components if there is an edge labelled g_i going between

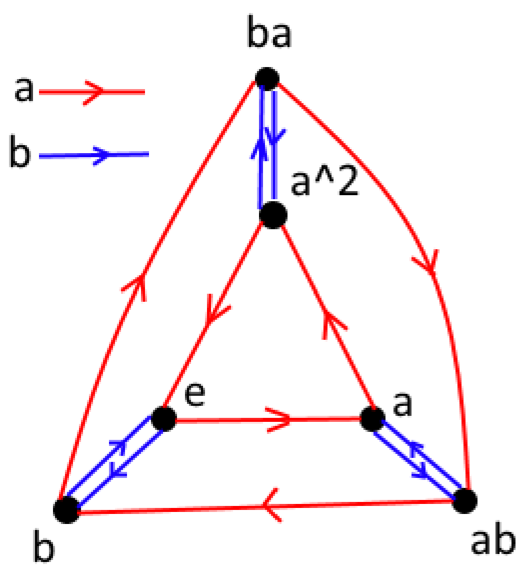


Figure 7: An example of a Cayley graph

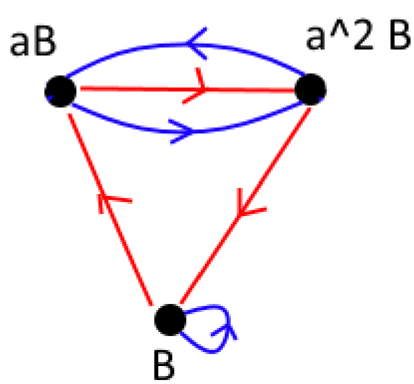
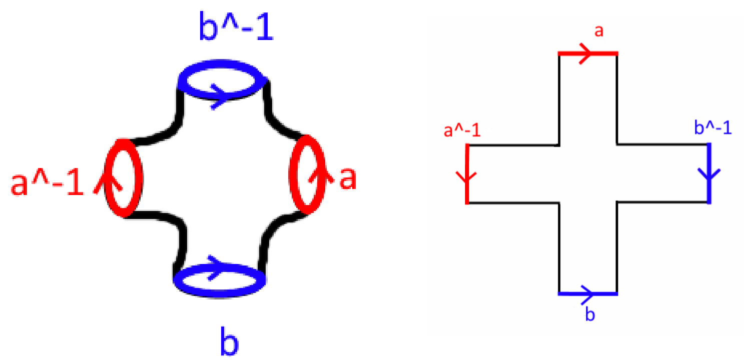


Figure 8: A relative Cayley graph



(a) Without boundary

(b) With boundary

Figure 9: Building blocks for surfaces

the corresponding vertices of the Cayley graph. The component marked g_i at the source of the arrow is attached to the component marked g_i^{-1} at the target of the arrow.

Figure 9 depicts two possible choices of building blocks, with marked boundary components identified. The cross-shaped building block will create a surface with boundary, whereas the tube-shaped block will create a surface with no boundary.

In figures 10 and 11 we depict this process of creating surfaces, using the cross-shaped and tube-shaped building blocks respectively.

Consider the topology of these surfaces. The topology of an oriented surface is determined entirely by the its genus and the number of holes. What will the genus of our surfaces be? We can find it by computing the Euler characteristic of the "building blocks". Consider the decomposition shown in figure 12a. Each building block has 5 vertices, 8 edges, and 1 face. But note that each of the 4 vertices and edges on the boundary will be counted twice, because they will be merged when the surface is constructed. Therefore, in the completed surface there will be 3 vertices, 6 edges and 1 face per building block. A graph with n vertices will give rise to a surface with Euler characteristic $n(3-6+1) = -2n$. Since the genus g of a closed surface is related to the Euler characteristic χ by $\chi = 2-2g$, we have $g = -\frac{1}{2}(\chi - 2) = -\frac{1}{2}(-2n-2) = n+1$.

For the cross shaped building blocks, there is an easy cell decomposition

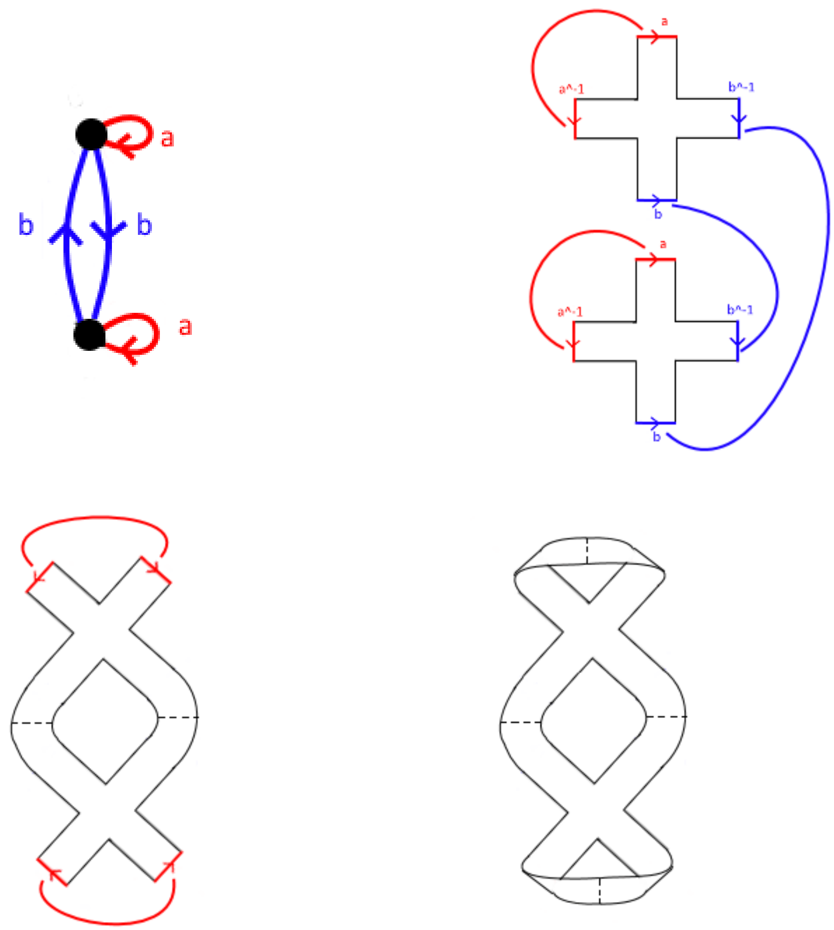


Figure 10: Constructing Surfaces from Cayley graphs – crosses

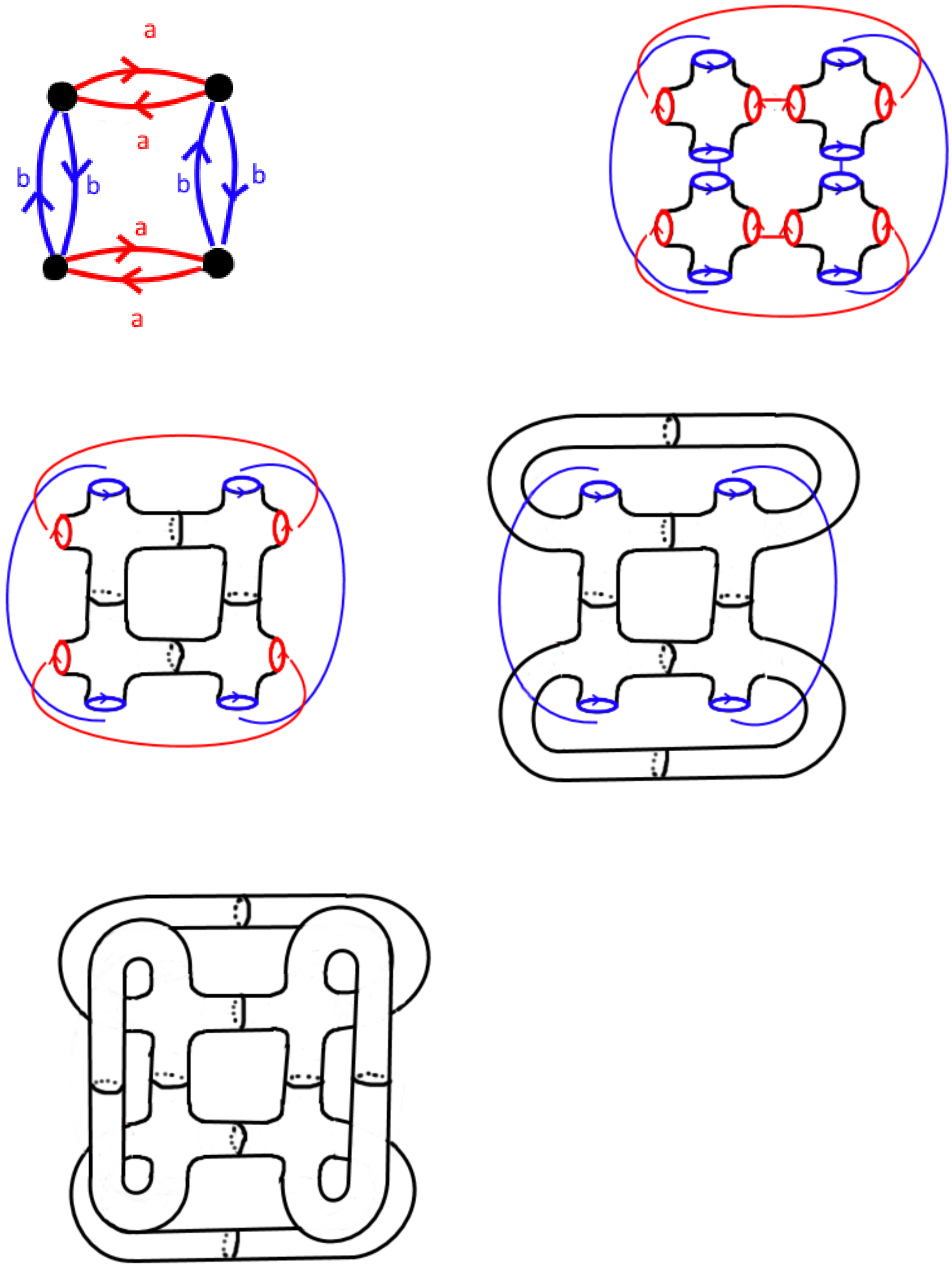
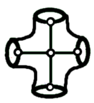
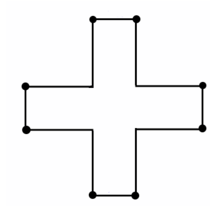


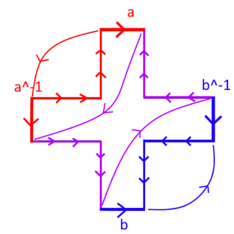
Figure 11: Constructing surfaces from Cayley graphs – tubes



(a) Tube decomposition



(b) Cross decomposition



(c) Cross boundary components

formed by placing a vertex at each exterior corner, seen in Figure 12b. Each building block has 1 face, 8 vertices, and 8 edges. The 4 edges and 8 vertices on the connecting boundary components are counted twice, so overall our surface will have Euler characteristic $\chi = n(1 - 6 + 4) = -n$.

To classify the surface topologically, we must also find the number of boundary components. To find this, we must consider the boundary components of the building block and how they are glued together. As seen in figure 12c, the boundary components can be classified into 3 types, one of which always follows the A arrows, one of which follows the B arrows, and one of which follows the A and then B arrows. We can find the number of boundary components by following the action of A, B, and AB on the vertices. Vertices are split into equivalence classes by which "A"-cycle, "B"-cycle or "AB"-cycle they are in. The number of boundary components is equal to the sum of the size of each of these respective equivalence classes. Once we know the number of boundary components C, we can then find the genus: Because subtracting a disk from a surface lowers its Euler characteristic by one, we have $\chi = 2 - 2g - C$, so $g = -\frac{1}{2}(\chi - 2 + C) = \frac{1}{2}(n + 2 - C)$.

There is a natural action of G on Cayley-graph-derived surfaces. Namely, each building block is translated by the action of G on the graph. This preserves the connection of handles because they are attached by the action of G.

We can use this action to automatically create a pair of isospectral surfaces for any finite group G and subgroups H_1, H_2 satisfying Sunada's condition. Here is how we do it. We create 3 graphs – the Cayley graph of G, and the relative Cayley graphs with subgroups H_1, H_2. In each case we use the right G-action on itself and on the cosets. We now create 3 surfaces modelled on these Cayley graphs, S_G, S_{H_1}, and S_{H_2}. From our construction of S_G, it follows that S_{H_1} and S_{H_2} are the quotients of S_G with respect to

H_1 and H_2. Therefore, if H_1 and H_2 satisfy Sunada's condition, S_{H_1} and S_{H_2} will be isospectral.

3.4 Isospectral Surfaces

Now we can put the pieces together and actually construct some isospectral surfaces using Sunada's method. The group we use will be the smallest $SL(n,q)$ group, $SL(3,2)$. This example was given in [4].

First, we must choose generators for $SL(3,2)$ to construct the Cayley graph. Two elements that work are

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Lemma 3. *A and B generate $SL(3,2)$.*

Proof. All elements of $SL(3,2)$ can be written as products of elementary matrices – matrices which swap two rows or add one row to another. So it suffices to express all such elementary matrices in terms of A and B.

First, note that

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which is a cyclic permutation of the rows. Now,

$$A^{-1}B^{-1}AB^{-1}AABA^{-1}B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which swaps rows two and three. These two matrices are sufficient to arbitrarily permute the rows. Then we have

$$A^{-1}B^{-1}A^{-1}B^{-1}ABBAAB = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which adds row 2 to row 1. Combined with the ability to arbitrarily permute the rows, this allows us to add any row to any other row. So we can generate any elementary matrix with A and B, and therefore all of $SL(3, 2)$.

□

To construct the reduced Cayley graphs on G/H_1 and G/H_2, we must find how A, B, and their inverses act on the G/H_1 and G/H_2. The following lemma lets us easily compute these actions.

Lemma 4. *The left cosets of H_1 and H_2 are in one-to-one correspondence with the points and lines of $\mathbb{F}_2^3$. The action of elements of G on the cosets is the same as their action on these points and lines.*

Proof. H_1 and H_2 are the stabilizers of a point and a line, respectively. Consider a coset aH_1. The vector e_1 will be sent to $a(e_1)$ by any element of this coset, because ah_1e_1, by the definition of H_1. This shows that ae_1 is the same regardless of what representative a of H_1 we choose. If two cosets aH_1 and bH_1 have $ae_1 = be_1$, this implies $a^{-1}be_1 = e_1$. So $a^{-1}b \in H_1$, meaning they lie in the same coset. This shows that a coset aH_1 is uniquely determined by $a(e_1)$.

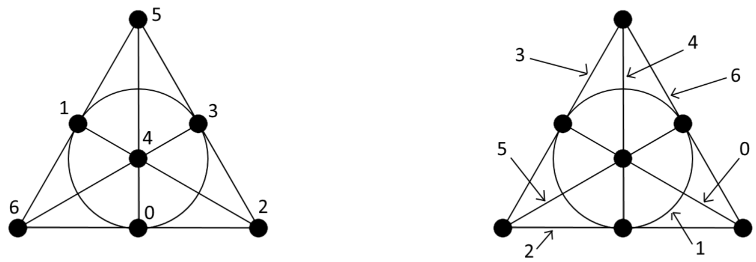


Figure 13: A numbering of the points and lines of the Fano Plane

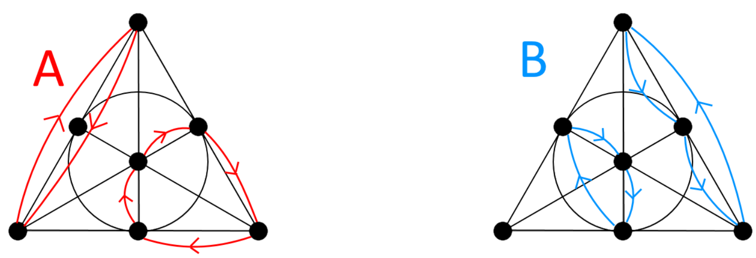


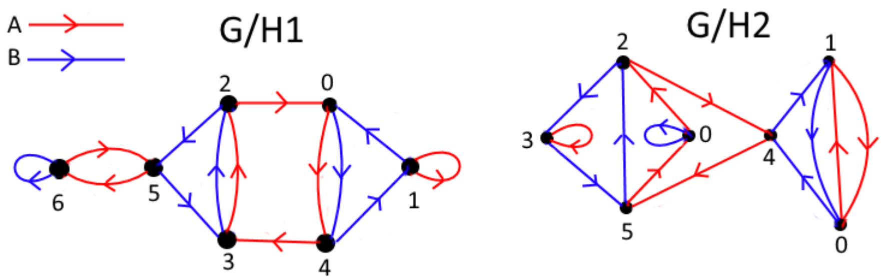
Figure 14: The action of our generators on the Fano Plane

Next we wish to show that the left action of G on these cosets is the same as their action on the points of $\mathbb{F}_2^3$. But this is fairly clear, as $a(bH_1) = (ab)H_1$, so the coset represented by ae_1 is sent to $b(ae_1)$.

The above argument only used the fact that H_1 was a stabilizer of points, so the same proof implies that left cosets of H_2 are in correspondence with lines of $\mathbb{F}_2^3$, and that the action of G on these cosets is the same as the action of G on the lines. $\square$

It's easy to compute the action of A and B on the points and lines of $\mathbb{F}_2^3$. Figure 14 shows their action on the Fano plane.

We can read off the action on the points and lines of $\mathbb{F}_2^3$ from these diagrams. They are:



	x	0	1	2	3	4	5	6
Points	$A(x)$	4	1	0	2	3	6	5
	$B(x)$	1	4	5	2	0	3	6

	y	0	1	2	3	4	5	6
Lines	$A(y)$	1	0	4	3	5	6	2
	$B(y)$	4	0	3	5	1	2	6

From these Cayley graphs we can deduce the structure of the corresponding surfaces. Using the tubular building block, we obtain two genus 8 closed surfaces. For the cross-shaped building block, we can count the number of A-orbits, B-orbits and AB-orbits. This can be done fairly easily – there are 3 of each kind of orbit in each graph. Therefore, there are 9 boundary components. The Euler characteristic is $-n = -7$, so our genus is $\frac{1}{2}(7 + 2 - 9) = 0$. Therefore, we have produced two 9-holed spheres.

These surfaces are isospectral – but we'd like them to be non-isometric. Are they? In the case of the cross-shaped building block, it's fairly easy to see that they are. Note that the center of each cross is a "special point" on that cross, because it's equidistant from the 4 inner corners. So central points of crosses must be sent to central points by isometries. Similarly, corners must be sent to corners. If we give a flat metric to the cross, we can pretend that each building block is a region of the plane. By the rigidity of planar congruences, the images of these 5 points determine the image of the entire block. This means that building blocks are sent to building blocks, so an isometry of the surfaces induces a graph isomorphism between the relative Cayley graphs of G/H_1 and G/H_2. But it's clear that no such isomorphism exists: note that in both graphs, there are two vertices with a single self-loop. However, in G/H_1 the distance between these 2 vertices is 4, and in G/H_2 the distance if 2. So our surfaces are non-isometric but isospectral!

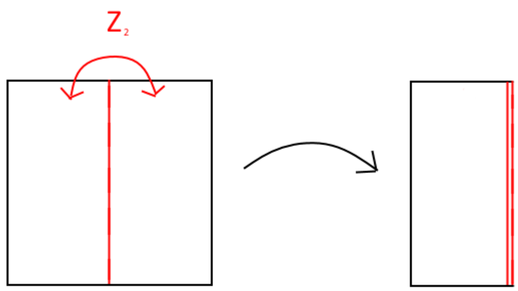


Figure 16: Quotient by Reflection

In the case of the tubular building blocks, an analogous argument based on "drawing the graph" on the surface will work. However, the full proof is rather technical and involves a great deal of hyperbolic geometry. The interested reader is referred to [3].

4 Transplantation via Sunada's Method

The method of transplantation, explained in section 3, can be derived using Sunada's method. The two triangular domains used can be obtained as the quotients of a 2-dimensional space under the action of a $SL(3, 2)$, and can be constructed from relative Cayley graphs just like the isospectral surfaces from the last section. This section will sketch how this can be done, first explaining the building blocks used and how they are pasted together, then the generating set of $SL(3, 2)$ used. It will exhibit the Cayley graphs obtained from H_1 and H_2 and how these lead to the earlier-shown triangular isospectral plane domains.

Our method of constructing isospectral surfaces with cayley graphs has only produced closed surfaces, not the planar domains which were our original goal. The method can be adapted by using reflections, instead of translations, as the action of our generators on the basic building blocks. The quotient of a plane domain by a reflection is shown in figure 16.

Taking a quotient by a reflection produces a boundary where there was none before. We can use this fact to obtain planar domains: by taking quotients of reflections, we can convert a boundaryless surface into one with boundary, which can then by embedded in $\mathbb{R}^2$.

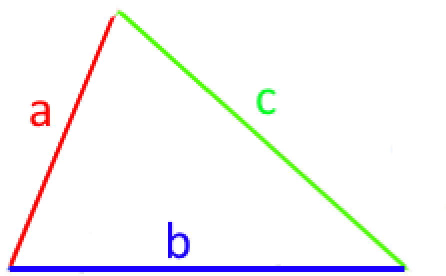


Figure 17: Our building block

We'll construct our domain out of the simplest possible 2 dimensional shape, the triangle. We'll use as building block a scalene acute-angled triangle. Since the triangle has 3 sides, we will need a group generated by 3 elements of order 2. In figure 17 we depict the identification of sides we will use. Our building blocks are glued together along sides of matching lengths, such that each block is the reflection of the triangles adjacent to it.

For our group G we can use $SL(3,2)$ again. Technically speaking, using $SL(3,2)$ means that our covering manifold M will not be a manifold at the corners where triangles meet, because the total angle there will exceed 2π. For instance, the element ab has order 4, implying that there will be 8 triangles around a single vertex. But this will not affect the application of Sunada's theorem, since there are only finitely many such points, all of which will be sent to the boundary by covering maps.

For our generating set, we'll use the following three elements:

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The effect of these generators on the Fano plane is depicted in figure 18. For our subgroups satisfying the Sunada condition, we once again use H_1 and H_2

Reading off the action on the points and lines(using the same numbering as in the last section), we obtain:

	x	0	1	2	3	4	5	6
Points	$a(x)$	1	0	5	3	4	2	6
	$b(x)$	2	1	0	4	3	5	6
	$c(x)$	4	6	2	3	0	5	1

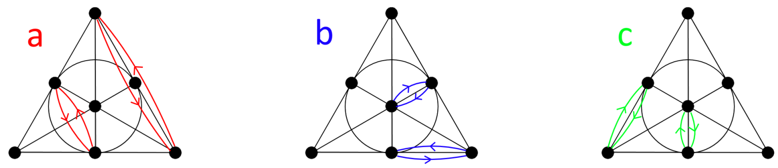


Figure 18: The action of the generators a,b,c on the Fano plane

	y	0	1	2	3	4	5	6
Lines	$a(y)$	4	1	3	2	0	5	6
	$b(y)$	1	0	2	3	6	5	4
	$c(y)$	2	5	0	3	4	1	6

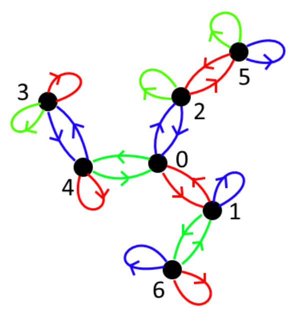
From these actions we obtain the Cayley graphs depicted in figure 19. From these Cayley graphs and the given triangle we obtain the isospectral domains we exhibited in section 2.

On its own this does not show on its own that the two domains are Dirichlet isospectral. In fact, it shows that they are Neumann isospectral, because the eigenfunctions continue into themselves across the reflecting boundary arcs. However, once we have obtained the domains, the proof by transplantation easily shows them to be Dirichlet isospectral as well.

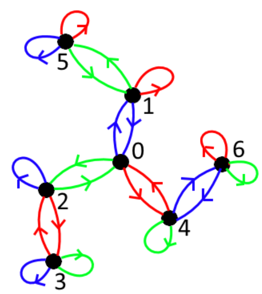
This example was originally obtained using a group of symmetries of a tiling of the hyperbolic plane. A homomorphism was constructed from this group to $SL(3, 2)$, and the resulting hyperbolic triangles were deformed into Euclidean ones. This is a fascinating topic, but is beyond the scope of this thesis. The interested reader is referred to [6] and section 4 of the original paper [5].

5 Conclusion

This thesis has presented examples of isospectral plane domains and shown how to prove that they are isospectral but non-congruent using transplantation. It has also considered a more general technique for constructing isospectral manifolds, Sunada's method, and shown how the simple transplantation-based examples were arrived at using this method. It is hoped that this thesis will be helpful to anyone who wishes to understand how isospectral non-congruent domains can be constructed.



(a) Relative Cayley graph of H_1



(b) Relative Cayley graph of H_2

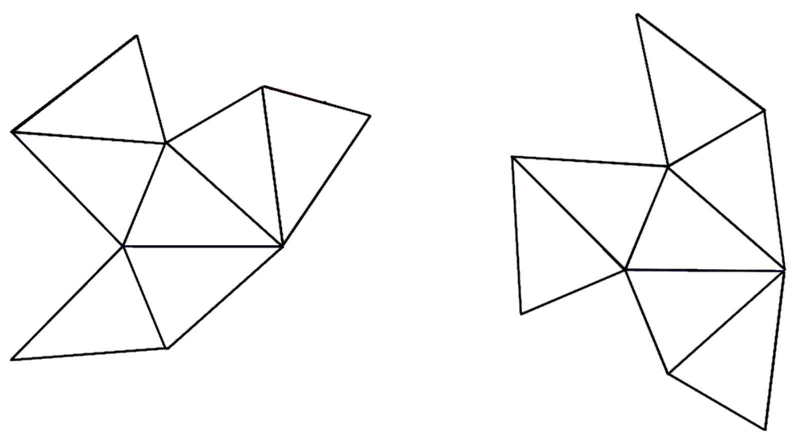


Figure 19: Relative Cayley graphs with our new generators and resulting domains

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